–HAVOC–
Here's another Voronoi code

Computational Physics Tübingen

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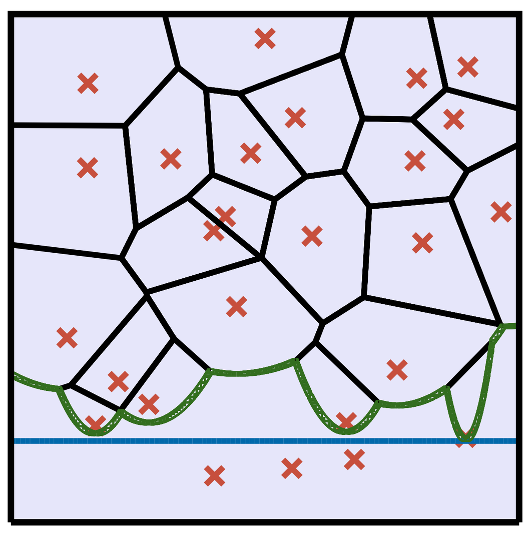
Overview

We have implemented a Voronoi code for solving the two-dimensional Euler equations. In this approach, a Voronoi tesselation is calculated from a cloud of mesh generating points in every time step. The Euler equations are then solved on the grid provided by the tesselation. At the end of each time step, the particles are moved according to the velocity field of the fluid. We simulated several test problems and compared the Lagrangian version of our code (moving particles) to the Eulerian version (fixed particles, grid code).

Fortune's Sweep Line Algorithm



The sweep line (blue) moves across the box from top to bottom and creates the Voronoi tesselation (black) from the mesh generating sites (red). Points in space above the beach line (green) already can be assigned to sites. This line consists of parabolas whose intersections trace the edges of the Voronoi tesselation.

Solving the Euler Equations

We use Springel (2010) as a guide for solving the Euler equations

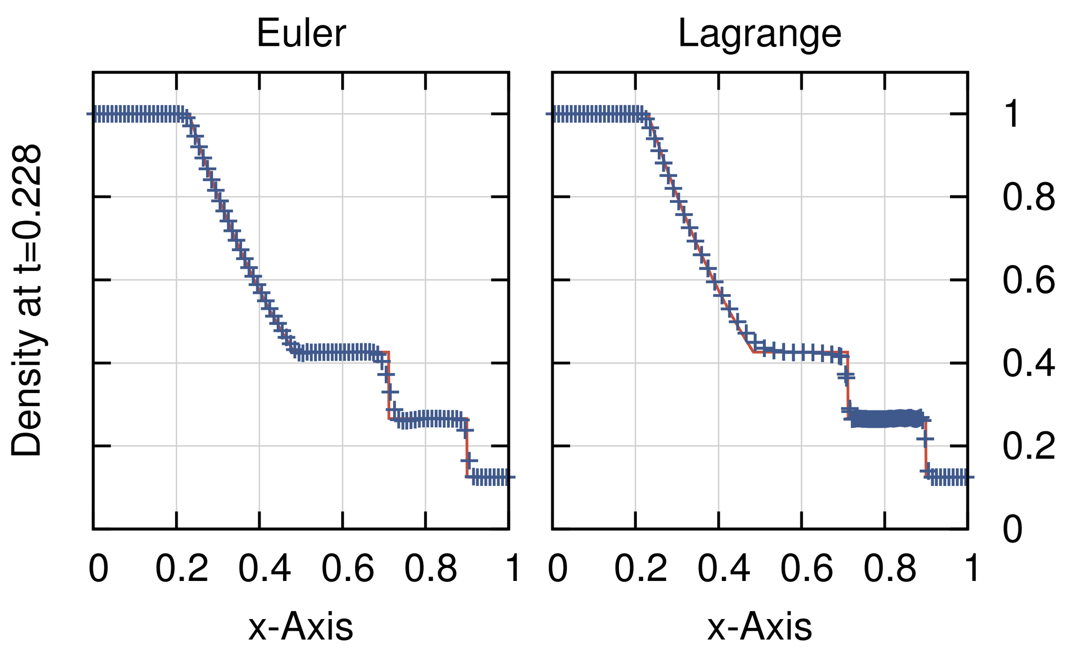
$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}(\rho \boldsymbol{v}) = 0,$$
$$\frac{\partial \rho \boldsymbol{v}}{\partial t} + \boldsymbol{\nabla}\left(\rho \boldsymbol{v}\boldsymbol{v}^T\right) + \boldsymbol{\nabla} P = \boldsymbol{0},$$
$$\frac{\partial \rho e}{\partial t} + \boldsymbol{\nabla}\left((\rho e + P)\boldsymbol{v}\right) = 0.$$

- Godunov finite-volume method
- Second order in space and time
- Exact Riemann solver (Toro, 2009)

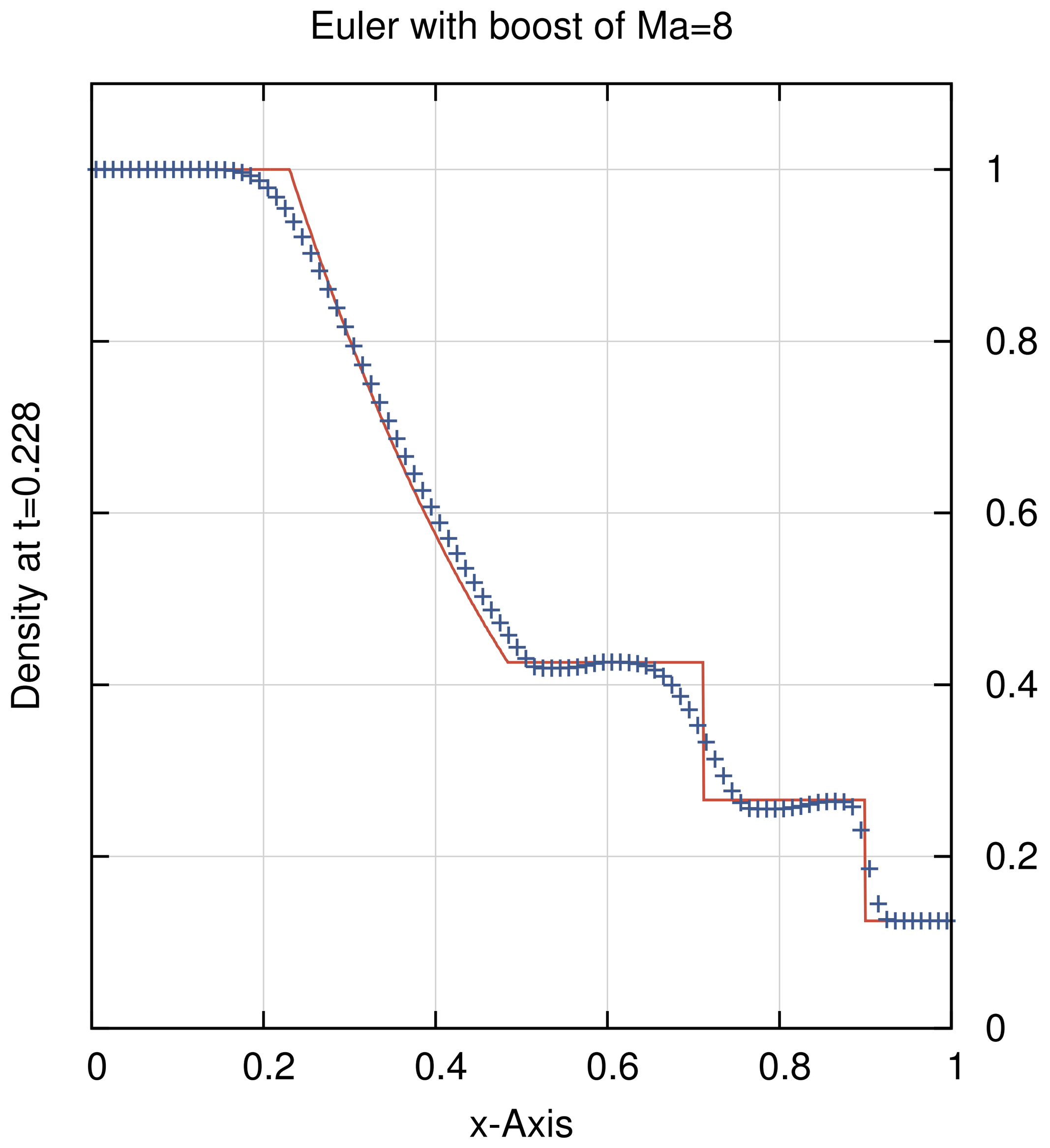
Shock Tube

Resolution: 100×10 particles
Domain: $(x, y) \in [0, 1] \times [-0.05, 0.05]$

$\gamma = 7/5$	Left state	Right state
Pressure	$P_l = 1$	$P_r = 0.1$
Density	$\rho_l = 1$	$\rho_r = 0.125$
Velocity	$v_l = 0$	$v_r = 0$

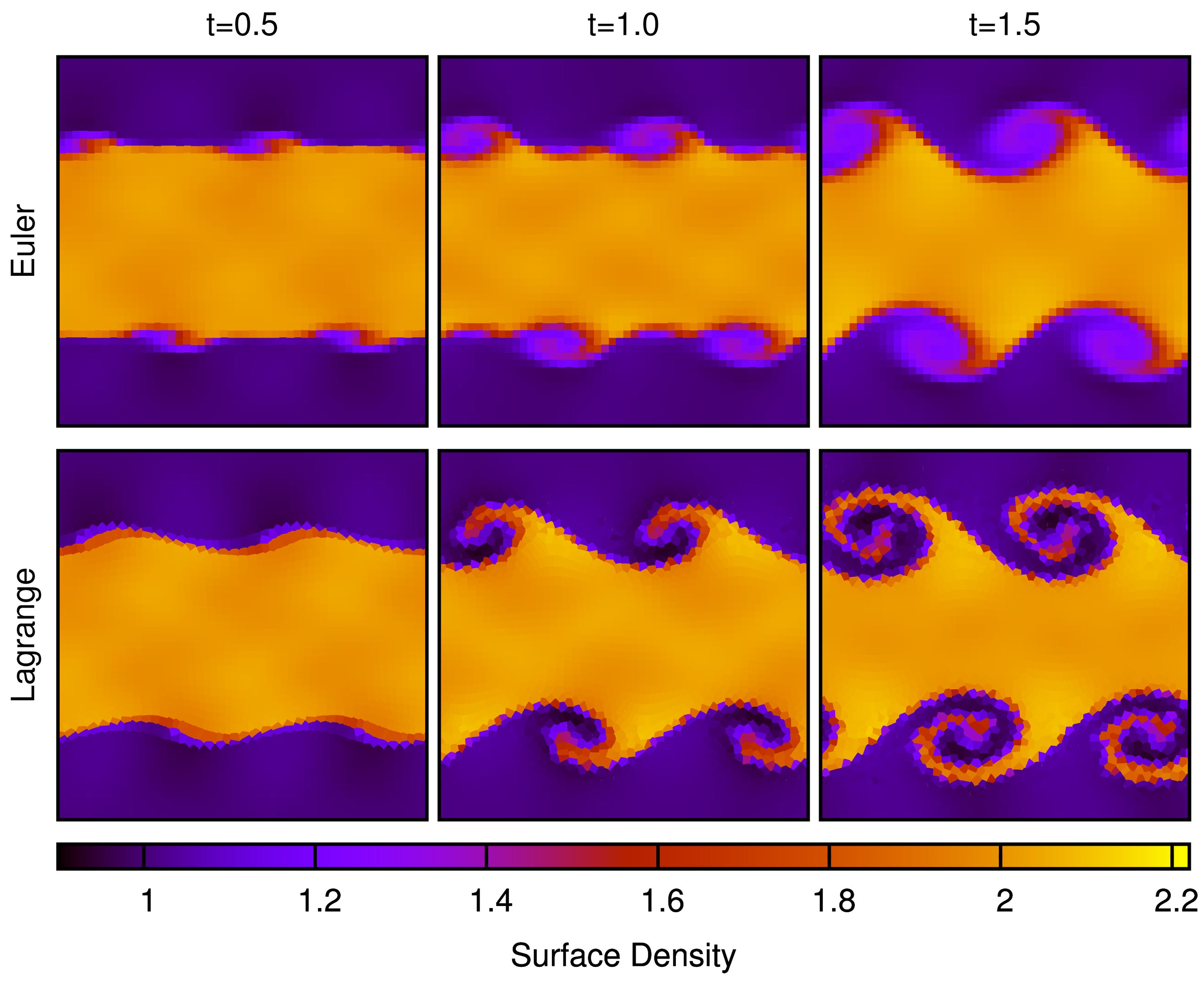


We show the lack of Galilean invariance of the Euler code by adding a boost velocity of Mach 8 to v_l and v_r. The outcome can be seen below:



The Lagrangian version of the code is of course not affected by a boost velocity.

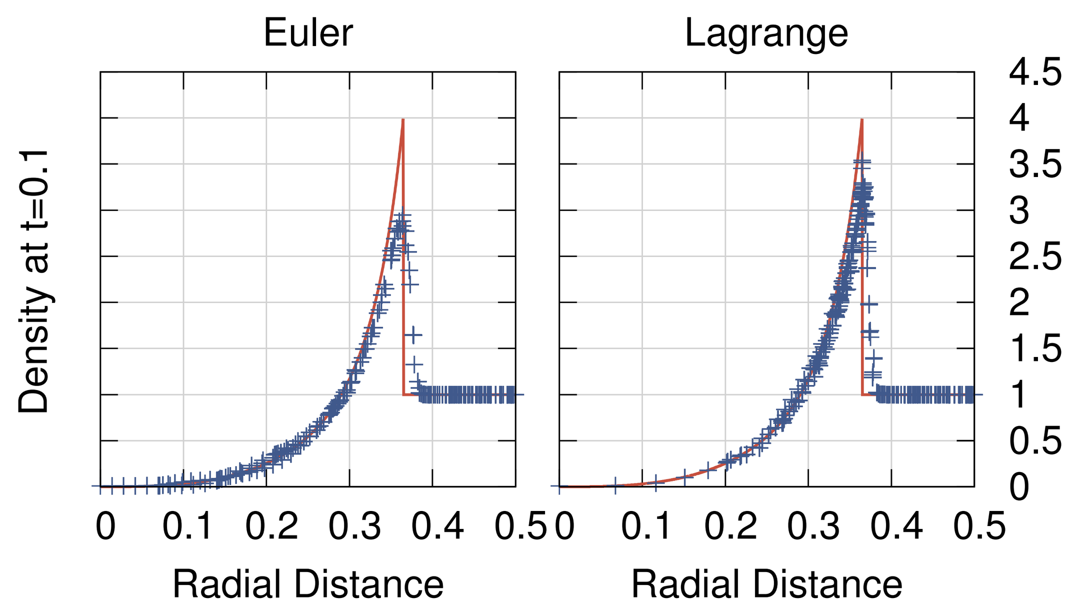
Kelvin-Helmholtz Instability



50×50 particles, $(x, y) \in [-0.5, 0.5] \times [-0.5, 0.5]$, $\rho_{\text{middle}} = 2$, $\rho_{\text{outside}} = 1$, $P = 2.5$, $v_{\text{rel}} = 1$, $\gamma = 5/3$. Furthermore, in order to trigger a single mode, the vertical velocity is slightly perturbed with a sine function. The billows of the Lagrangian simulation are sharper because of much less mixing. Moreover, the instability evolves slightly faster in the Voronoi approach.

Sedov Blast Wave

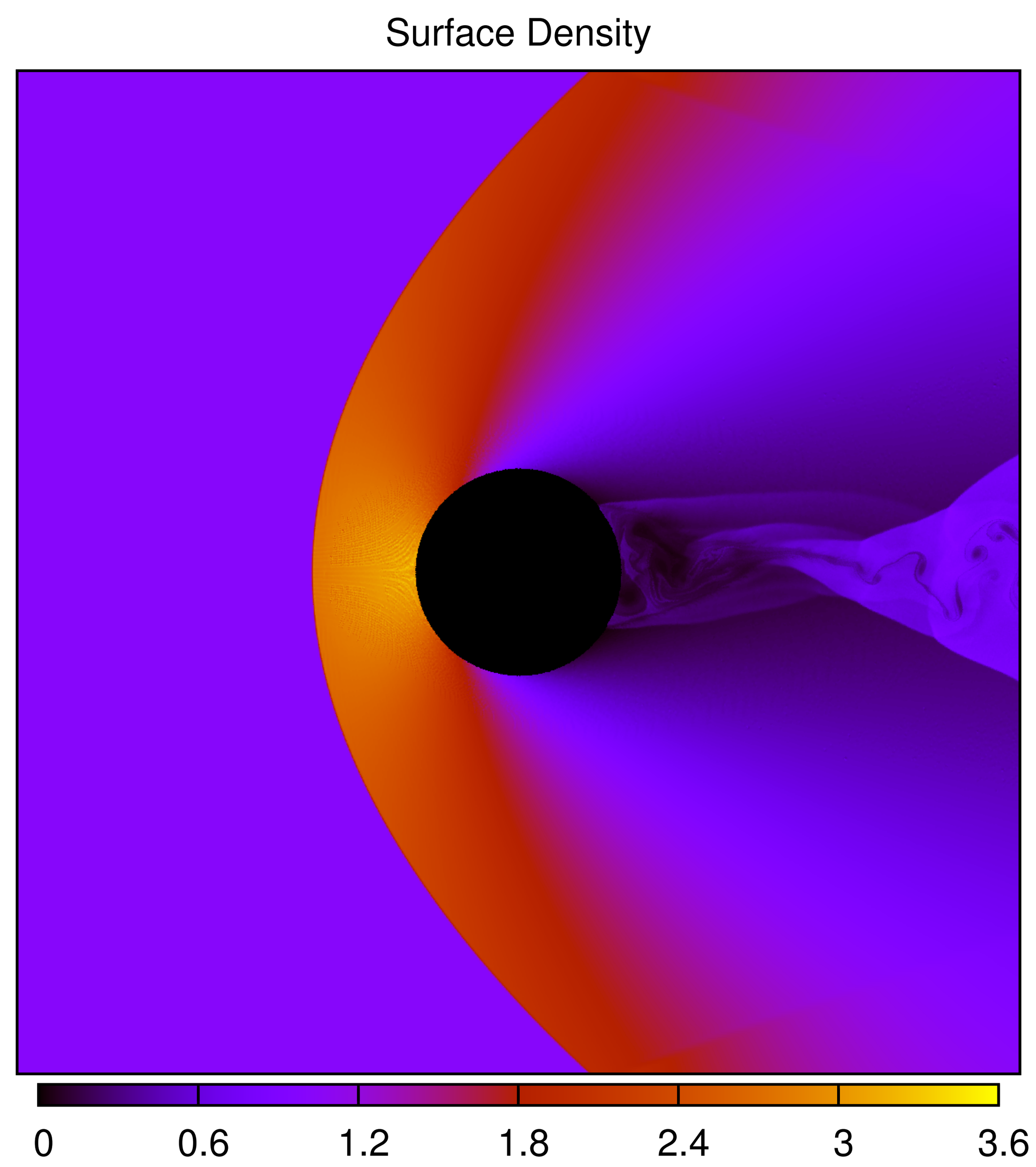
Resolution: 101×101 particles
Domain: $(x, y) \in [-0.5, 0.5] \times [-0.5, 0.5]$
$E_{\text{injected}} = 1$, $\rho_{\text{ini}} = 1$, $\gamma = 5/3$.



The Lagrangian version retains constant mass per cell and therefore reproduces the sharp density spike better.

Wind Tunnel

A 2D-sphere at Mach 2 and $\gamma = 7/5$. The initial resolution is 500×500 particles and we apply dynamic mesh refinement.



Conclusions

- \+ Galilean invariance
- \+ Constant mass per cell
- \+ Only little mixing
- \+ Grid based (well known solvers)
- \+ Easy implementation of obstacles
- − High numerical cost
- \+ Voronoi is fun :)

Media and Contact

You may find movies of the sweep line algorithm and the simulations on my YouTube channel.

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References

De Berg, M., O. Cheong, M. Van Kreveld, and M. Overmars 2008. *Computational Geometry: Algorithms and Applications*. Springer.

Fortune, S. 1987. A sweepline algorithm for voronoi diagrams. *Algorithmica* **2**, 153–174.

Springel, V. 2010. E pur si muove: Galilean-invariant cosmological hydrodynamical simulations on a moving mesh. *Royal Astronomical Society, Monthly Notices* **401**, 791–851.

Toro, E. 2009. *Riemann Solvers and Numerical Methods for Fluid Dynamics*. Springer London, Limited.