Overview
We have implemented a Voronoi code for solving the two-dimensional Euler equations. In this approach, a Voronoi tessellation is calculated from a cloud of mesh generating points in every time step. The Euler equations are then solved on the grid provided by the tessellation. At the end of each time step, the particles are moved according to the velocity field of the fluid. We simulated several test problems and compared the Lagrangian version of our code (moving particles) to the Eulerian version (fixed particles, grid code).

The Voronoi tessellation (black) from the mesh generating sites (red). Points in space above the beach line (green) already can be assigned to sites. This line consists of parabolas whose intersections trace the edges of the Voronoi tessellation.

Resolution: 100 × 10 particles
Domain: \((x, y) \in [0, 1] \times [-0.05, 0.05]\)
\(\gamma = 7/5\)

Left state
Pressure \(P_l = 1\)
Density \(\rho_l = 1\)
Velocity \(v_l = 0\)

Right state
Pressure \(P_r = 0.1\)
Density \(\rho_r = 0.125\)
Velocity \(v_r = 0\)

We show the lack of Galilean invariance of the Euler code by adding a boost velocity of Mach 8 to \(v_l\) and \(v_r\). The outcome can be seen below:

The Lagrangian version of the code is of course not affected by a boost velocity.

Fortune's Sweep Line Algorithm
The sweep line (blue) moves across the box from top to bottom and creates the Voronoi tessellation.

Resolution: 101 × 101 particles
Domain: \((x, y) \in [-0.5, 0.5] \times [-0.5, 0.5]\)
\(E_{\text{injected}} = 1\), \(\rho_{\text{ini}} = 1\), \(\gamma = 5/3\).

Solving the Euler Equations
We use Springel (2010) as a guide for solving the Euler equations
\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \nabla (\rho v) &= 0, \\
\frac{\partial \rho v}{\partial t} + \nabla (\rho v v^T) + \nabla P &= 0, \\
\frac{\partial \rho c}{\partial t} + \nabla ((\rho c + P)v) &= 0.
\end{align*}
\]
• Godunov finite-volume method
• Second order in space and time
• Exact Riemann solver (Toro, 2009)

Shock Tube

\(50 \times 50\) particles,
\((x, y) \in [-0.5, 0.5] \times [-0.5, 0.5]\),
\(\beta_{\text{middle}} = 2\), \(\beta_{\text{outside}} = 1\),
\(P = 2.5\), \(v_{\text{ini}} = 1\), \(\gamma = 5/3\).
Furthermore, in order to trigger a single mode, the vertical velocity is slightly perturbed with a sine function.
The billows of the Lagrangian simulation are sharper because of much less mixing. Moreover, the instability evolves slightly faster in the Voronoi approach.

Kelvin-Helmholtz Instability

Wind Tunnel
A 2D-sphere at Mach 2 and \(\gamma = 7/5\). The initial resolution is 500 × 500 particles and we apply dynamic mesh refinement.

Sedov Blast Wave

Media and Contact
You may find movies of the sweep line algorithm and the simulations on my YouTube channel.

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References


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